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NON-STATIONARY RANDOM VIBRATION OF A LINEAR STRUCTURE

P-T. D. SPANOS
Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, TX 78712, U.S.A.

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Abstract—The non-stationary random vibration of a *lightly damped* linear structure subjected to white noise is considered. It is shown that the probability density function of the amplitude of the structural response can be approximated by a Rayleigh distribution. Analytical formulae for the time dependent statistics of the amplitude are presented. The analytical results are compared with data obtained by a numerical simulation.

INTRODUCTION

The problem of the response of linear structures to random excitation has occupied the engineer and the applied scientist for quite some time. The theory for determining the statistical properties of the structural response has been quite well developed and is available in standard textbooks[1, 2]. However, problems pertaining to the statistical properties of the maximum of the response or to the time at which the response exceeds a certain barrier are still under investigation.

If the damping of the structural system is small, the response exhibits pseudo-sinusoidal behavior with slowly varying, in time, amplitude and phase. Evidently, the determination of the statistics of the response amplitude is particularly important in estimating the failure potential of the dynamical system, which essentially is the goal of a probabilistic analysis of a physical problem. It is understood that the concept of the response amplitude is applicable only when the damping of the system is small. However, this assumption can be justified for a large class of physical systems of engineering interest.

If it is desired to determine the statistics of the response amplitude at a time much longer than the rise time of the structure, then the response can be assumed to be stationary. Under this assumption, it can be shown that the probability density function of the response amplitude is a Rayleigh distribution[2]. The crucial point of the proof is that the response being stationary, the correlation of the response and its time derivative is zero. Clearly, this is not true if the process is non-stationary. This will be the nature of the response at times which are much shorter than the rise time of the structure. The problem of the non-stationary random response of a linear single-degree-of-freedom system to white noise has been originally examined in Ref. [3]. Typical examples of research effort pertinent to this problem are given in Refs. [4–6].

In the present paper, an approximate probability density function for the *amplitude* of the non-stationary response of a *lightly* damped linear structure is derived. Analytical formulae for the time-dependent moments are given. The results of the analytical approach are compared with data obtained by a numerical simulation.

MATHEMATICAL BACKGROUND

Consider a linear single-degree-of-freedom structure described by the stochastic differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = w(t), \tag{1}$$

where ω_n is the natural angular frequency, and ζ is the ratio of critical damping of the structure. The symbol $w(t)$ represents a white noise process with spectral density S constant over the interval $(-\infty, \infty)$. The dot above a variable denotes differentiation with respect to time.

Define the processes $a(t)$ and $\phi(t)$ by

$$x(t) = a(t)\cos(\omega_n t + \phi(t)) \tag{2}$$

and

$$\dot{x}(t) = -\omega_n a(t)\sin(\omega_n t + \phi(t)). \tag{3}$$

Using eqns (2) and (3) it is readily proved that

$$a^2(t) = x^2 + \dot{x}^2/\omega_n^2 \tag{4}$$

and

$$\phi(t) = -\tan^{-1}\left(\frac{\dot{x}}{\omega_n x}\right) - \omega_n t. \tag{5}$$

Using eqns (4) and (5) differential equations governing $a(t)$ and $\phi(t)$ can be derived as follows. Differentiating eqns (4) and (5) with respect to time and using eqn (1) it is found

$$\dot{a}(t) = -2\zeta\omega_n a\sin^2(\omega_n t + \phi(t)) - \frac{w(t)}{\omega_n}\sin(\omega_n t + \phi(t)) \tag{6}$$

and

$$\dot{\phi}(t) = -\frac{2}{a}\zeta\omega_n\sin(\omega_n t + \phi(t))\cos(\omega_n t + \phi(t)) - \frac{w(t)}{\omega_n a}\cos(\omega_n t + \phi(t)). \tag{7}$$

At this stage, additional assumptions about the problem are made. It is assumed that the damping of the structure and the spectral density of the excitation are small. Mathematically, these assumptions may be described by

$$\zeta \ll 1 \tag{8}$$

and

$$S = 0(\zeta). \tag{9}$$

Under these assumptions, it can be argued that $a(t)$ and $\phi(t)$ are slowly varying functions of t. Therefore, the response $x(t)$, eqn (2), exhibits pseudo-sinusoidal behavior like the one shown in Fig. 1.

Using an asymptotic method introduced by Stratanovich[7], eqns (6) and (7) can be used to derive the following equation for the amplitude $a(t)$[8–10]

$$\dot{a} = -\zeta\omega_n\left(a - \frac{\sigma^2}{a}\right) + (2\zeta\omega_n\sigma^2)^{1/2}\eta(t), \tag{10}$$

where

$$\sigma^2 = \frac{\pi S}{2\zeta\omega_n^3} \tag{11}$$

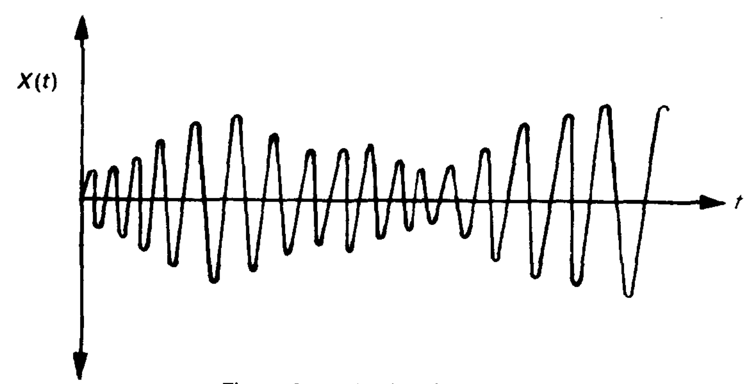

Fig. 1. Narrow-band random process.

is the stationary variance of $x(t)$, and $\eta(t)$ is a zero-mean, delta-correlated process with unit intensity, i.e. $E[\eta(t)\eta(t+\tau)] = \delta(\tau)$. In deriving eqn (10), various rapidly oscillating terms are first averaged over one cycle, $T = 2\pi/\omega_n$, of oscillation in a way similar to that of the technique of Bogoliubov and Mitropolski[11]. Next, the fact that the band-width of the excitation is much greater than that of the response, is exploited to justify the assumption of statistical independence of $w(t)$ and the values of $a(t)$ and $\phi(t)$ corresponding to a slightly shifted time $t \pm \Delta t$.

Equation (10) describes approximately the random evolution, in time, of the amplitude. Its important feature is that it is not coupled with the phase $\phi(t)$. This fact simplifies significantly the study of the random behavior of the amplitude. For example, in Ref. [8], eqn (10) has been used to derive ordinary differential equations governing the moments of the first passage time of the amplitude. In addition, the Kolmogorov equation associated with eqn (10) has been used in Ref. [10] to study the probability distribution function of the first passage time of the amplitude. In this paper, eqn (10) will be used to determine the statistics of the amplitude.

AMPLITUDE STATISTICS

The Fokker–Planck equation associated with the stochastic differential eqn (10) is

$$\frac{\partial p(a,t)}{\partial t} = \frac{\partial}{\partial a}\left[\zeta\omega_n\left(a - \frac{\sigma^2}{a}\right)p\right] + \zeta\omega_n\sigma^2\frac{\partial^2 p}{\partial a^2}. \tag{12}$$

The symbol $p(a,t)$ represents the probability density function of $a(t)$. It is assumed that the structure is initially at rest. Probabilistically, this can be expressed as

$$p(a,0) = \hat{\delta}(a), \tag{13}$$

where $\hat{\delta}(a)$ represents the one-sided Dirac delta function. The boundary conditions for the partial differential eqn (12) can be determined by imposing restrictions on $a(t)$. Since $a(t)$ represents the amplitude of $x(t)$, and therefore is non-negative, it is reasonable to construct a solution of eqn (12) compatible with the restriction $0 \le a \le \infty$. Under this restriction, the eigenvalues d_r and the eigenfunction $E_r(a)$ are found to be

$$d_r = 2\zeta\omega_n r, \qquad r = 0, 1, \ldots \tag{14}$$

and

$$E_r(a) = \frac{1}{r!}\frac{a}{\sigma^2}\exp(-a^2/2\sigma^2)L_r(a^2/2\sigma^2), \qquad r = 0, 1, \ldots \tag{15}$$

where L_r is the Laguerre polynomials. Using the properties of the Laguerre polynomials it can be readily proved that

$$\int_0^\infty \frac{E_m(a)E_r(a)}{E_0(a)}\,\mathrm{d}a = \delta_{mr}, \tag{16}$$

where δ_{mr} is the Kronecker delta symbol. The solution of eqn (12) can be put in the form

$$p(a,t) = \sum_{r=0}^{\infty} C_r \exp(-d_r t)E_r(a), \tag{17}$$

where C_r are constant coefficients to be determined by using the initial condition, eqn (13). Combining eqns (13) and (17) yields

$$p(a,0) = \hat{\delta}(a) = \sum_{r=0}^{\infty} C_r E_r(a). \tag{18}$$

Multiplying eqn (18) by E_r/E_0, integrating from zero to infinity, and using the orthonormality relation (16), it is found

$$C_r = 1; \qquad r = 0, 1, \ldots. \tag{19}$$

Therefore, eqn (18) can be rewritten as

$$p(a, t) = \sum_{r=0}^{\infty} \exp(-d_r t) E_r(a). \tag{20}$$

As $t \to \infty$, stationary solution, eqn (20) yields

$$\lim_{t\to\infty} p(a, t) = \frac{a^2}{\sigma} \exp(-a^2/2\sigma^2), \tag{21}$$

which is the Rayleigh density function. This function, governing the probability distribution of the amplitude of a stationary narrow-band Gaussian process, is usually derived by a quite different approach[2]. Guided by this result for the stationary solution, one might attempt to find a closed form solution for the non-stationary probability density function $p(a, t)$, eqn (20). This can be done by using the properties of the Laguerre polynomials. For the purpose, eqn (20) is rewritten as

$$p(a, t) = \frac{a}{\sigma^2} \exp(-a^2/2\sigma^2) \sum_{r=0}^{\infty} L_r(a^2/2\sigma^2)[\exp(-2\zeta\omega_n t)]^r. \tag{22}$$

It is known that[12]

$$\sum_{r=0}^{\infty} L_r(y) u^r = \frac{\exp(-yu/(1-y))}{1-u}. \tag{23}$$

Applying eqn (23) for $y = a^2/2\sigma^2$ and $u = \exp(-2\zeta\omega_n t)$, eqn (23) can be rewritten as

$$p(a, t) = \frac{a \exp\{-a^2/2\sigma^2[1 - \exp(-2\zeta\omega_n t)]\}}{\sigma^2[1 - \exp(-2\zeta\omega_n t)]}. \tag{24}$$

Therefore, it has been proved that the probability density function of the non-stationary response amplitude is approximately a time dependent Rayleigh distribution. This result, besides its theoretical significance, facilitates the determination of the statistical moments of the response amplitude $a(t)$. Specifically, using eqn (24) it can be readily proved that

$$E[(a/\sigma)^{2k+1}] = \sqrt{\left(\frac{\pi}{2}\right)} 2^{-k}(k+1)(k+2)\ldots(2k+1)[1 - \exp(-2\zeta\omega_n t)], \qquad k = 0, 1, \ldots \tag{25}$$

and

$$E[(a/\sigma)^{2k}] = 2^k k![1 - \exp(-2\zeta\omega_n t)]. \tag{26}$$

It is interesting to compare the results of the present analysis with the results of other approaches to similar problems. For example, eqn (26) can be used to find an approximate expression for the variable $E[x^2(t)]$. Specifically, using eqn (2) it can be easily verified that

$$x^2(t) = \frac{1}{2} a^2(t)[1 + \cos 2(\omega_\eta t + \phi(t))]. \tag{27}$$

If the rapidly oscillating term $\cos 2(\omega_\eta t + \phi(t))$ is neglected, as it was done for the derivation of eqn (11), and eqn (26) is applied for $k = 2$, it is found

$$E[x^2(t)/\sigma^2] = 1 - \exp(-2\zeta\omega_n t). \tag{28}$$

The exact expression for $E[x^2(t)]$ has been found in Ref. [3] by a different approach. The exact expression is

$$E[x^2(t)/\sigma^2] = 1 - \frac{\exp(-2\zeta\omega_n t)}{\omega_d^2}\left[\omega_d^2 + \omega_n\omega_d\zeta \sin(2\omega_d t) + \frac{(2\omega_n\zeta)^2}{2}\sin^2(\omega_d t)\right], \tag{29}$$

where $\omega_d^2 = \omega_n^2(1-\zeta^2)$. It is readily seen that eqn (29) reduces to eqn (28), if the rapidly oscillating term $\sin(2\omega_d t)$ and the $0(\zeta^2)$ term are neglected. It can be also seen that eqn (29) reduces to eqn (28) at times

$$t = t_k = \frac{\pi k}{\omega_d}; \qquad k = 0, 1, \ldots \tag{30}$$

independently of the assumption of small ratio of critical damping ζ.

The fact that the exact solutions for $E[x^2(t)]$, $E[\dot{x}^2(t)]$ and $E[x(t)\dot{x}(t)]$ are obtainable, with much calculational effort, however, could justify an alternate approach to the problem of the determination of $p(a, t)$ for the *special* case of *Gaussian* excitation. Specifically, assuming that $w(t)$ is Gaussian, it can be assured that $x(t)$ and $\dot{x}(t)$ are jointly Gaussian; the probability density function $p(x, \dot{x})$ will depend on $E[x^2(t)]$, $E[\dot{x}^2(t)]$ and $E[x(t)\dot{x}(t)]$. Subsequently, the probability density function $p(a, t)$ can be determined by using $p(x, \dot{x})$ and the algebraic transformation introduced by eqn (4). Using this approach it can be proved that at times specified by eqn (30), the probability density function given by eqn (24) is identical to the exact solution. This method, however, is applicable only for the case of Gaussian white noise and does not utilize simplifications justified by the small damping. In addition, this approach would be extremely cumbersome to apply to the problem of structural response to modulated Gaussian white noise because of the complexity of the corresponding solutions for $E[x^2(t)]$, $E[\dot{x}^2(t)]$ and $E[x(t)\dot{x}(t)]$. A typical example of these solutions is given in Ref. [13]. It is interesting to note that the methodology of the present paper can be readily applied to the above problem without requiring that the white noise excitation be Gaussian [14].

Equations (25) and (26) can also be used to determine the rise time of the structure. Define the rise time T_f as the time required for the structural response to reach a fraction f of its stationary level. Using eqns (25) and (26), it can be shown that the rise time for all moments of the amplitude is given by

$$\frac{T_f}{T} = \frac{\ln(1-f)}{4\pi\zeta}, \tag{31}$$

where $T = 2\pi/\omega_n$ is the natural period of the structure.

NUMERICAL RESULTS

For the purpose of checking the results of the present approximate analytical method, a simulation study of system (1) with $\zeta = 0.02$ was performed. For the generation of a sample function of the excitation $w(t)$, a sequence of normally distributed numbers $G_1, \ldots G_{200}$ was first generated. Subsequently, the values $G_1, \ldots G_{200}$ were assigned to 200 successive ordinates spaced at equal intervals $\Delta t' = 0.01$, along the dimensionless time abscissa $t' = t/T$. Linear variation of the ordinates over each interval was assumed. A complete ensemble of 250 such sample functions $x_r(t)$ $(r = 1, \ldots 250)$ was generated by repeating the above procedure 250 times. The response of the system (1) to each of the 250 sample function was computed by numerical integration. Subsequently, the non-stationary mean value $E(a)/\sigma$ and standard denotion $[E(a)^2 - E^2(a)]^{1/2}/\sigma$ of the amplitude were computed by averaging the numerical data. From Fig. 2 and Fig. 3 it is seen that the numerical data are in agreement with the corresponding analytical expressions for the mean value and the standard deviation of $a(t)$. It is noted that the analytical solution not only predicts the correct qualitative nature of the amplitude statistics, but the actual numerical values given by the two approaches are in close agreement in both the non-stationary and the stationary segments of the response.

The dimensionless rise time T_f/T, eqn (30), has been plotted in Fig. 4 vs the ratio ζ of critical

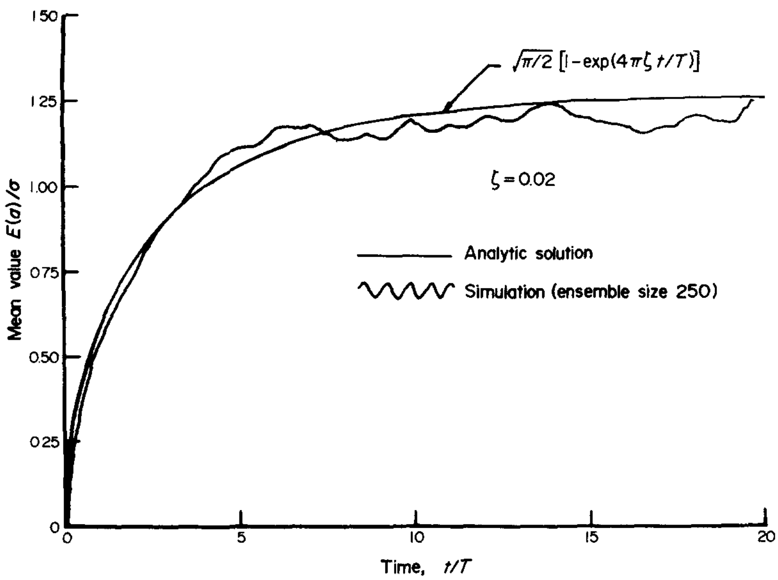

Fig. 2. Mean value of non-stationary response amplitude vs time.

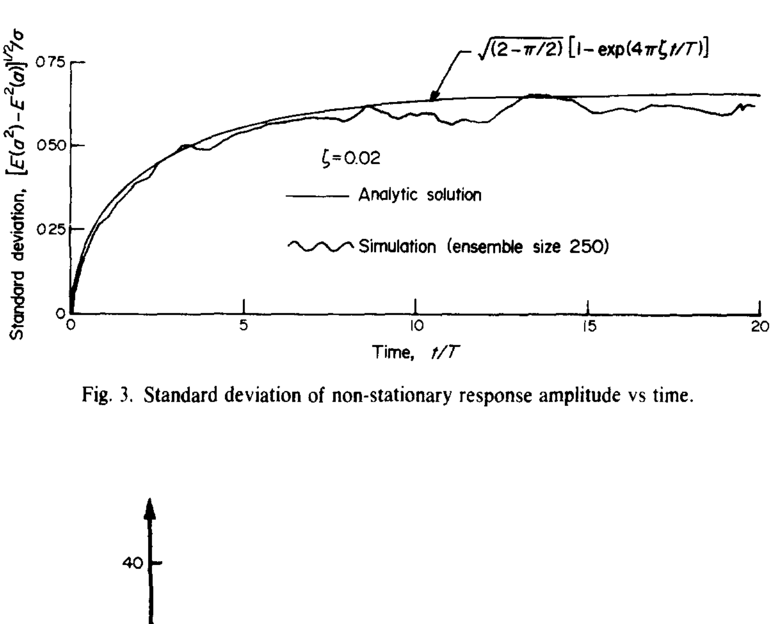

Fig. 3. Standard deviation of non-stationary response amplitude vs time.

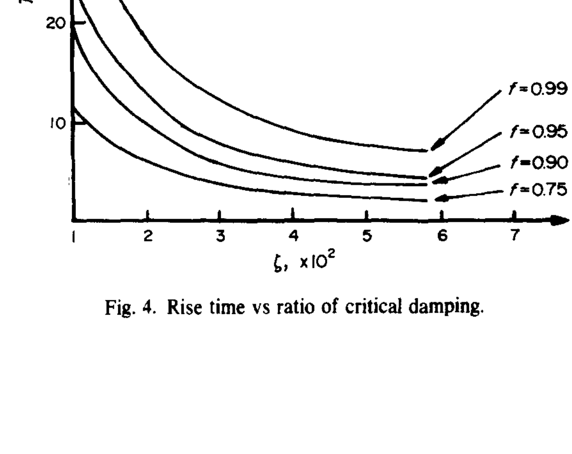

Fig. 4. Rise time vs ratio of critical damping.

damping. The fraction f of the stationary value of the structural response used for the definition of T_f has been selected as a parameter to identify each curve. From Fig. 3 the rise time of the structure corresponding to the given values of ζ and f can be read. For example, it is seen that for $\zeta = 0.01$ the structural response reaches 75% of its stationary level in a time approximately equal to 10 natural periods of oscillations, but approximately 25 more cycles of oscillation occur before the response reaches 99% of its stationary level.

SUMMARY

The statistical aspects of the amplitude of the non-stationary response of a *lightly* damped linear structure subjected to white noise excitation have been examined. It has been shown that the probability density function of the amplitude can be approximated by a time dependent Rayleigh ditribution. Analytical formulae for the statistical moments and the rise time of the response have been derived. The analytical results of the presented approach have been found in close agreement with the corresponding data of a numerical simulation study.

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